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The dynamics of homogeneous and symmetric cellular automata via description of neighbourhood distribution

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Abstract. The dynamics of cellular automata that are homogeneous and symmetric with respect to *up-down* symmetry is expressed by the probability of the appearance of different neighbourhoods on a lattice. The distribution function found in computer simulations is used to specify the differences in the set of cellular automata. The intrinsic structure of a rule has been proposed to explain the results obtained. The problem of whether or not automata are stable, the length of time needed to reach the stabilization and the type of stabilization, are also discussed.

1. Motivation

Generally, a cellular automata system consists of a *lattice*—the set of simple (with few levels) subsystems called spins $[\sigma]=\{\sigma_i\}$, and *rules*—the set of recipes, $R=\{r\}$, which give the method for finding the value of any spin in the next time step. As a result, cellular automata are the simplest models of discretized dynamics governed by a nonlinear equation. Evolution of such systems can be observed in step-by-step computer simulations. In the last few years cellular automata have been extensively studied [1, 2 and references quoted therein]. The model considered is popular and widely known and there exist several distinct approaches aimed at different applications.

In this paper a square lattice is considered with dimension L where every spin can be in one of two states: *up* $=1$, *down* $=0$: $\{\sigma_i=0,1;\ i=1,\ldots,L^2\}=[\sigma]_{L\times L}$. The adopted rules do not depend on a spin site i, and the state of any σ_i at time $t+1$ is determined by its nearest neighbourhood $\Theta_i(t)=(E_i(t),N_i(t),W_i(t),S_i(t))$,

$$\begin{array}{ccccc} * & \cdots & N_i & \cdots & * \\ \vdots & & \vdots & & \vdots \\ W_i & \cdots & \sigma_i & \cdots & E_i \\ \vdots & & \vdots & & \vdots \\ * & \cdots & S_i & \cdots & * \end{array} \tag{1.1}$$

in the following sense:

$$\sigma_i(t+1)=r(\Theta_i(t)). \tag{1.2}$$

Notice the correspondence between the notation in (1.1) and the geographic map's directions.

Since there is a great number of both the possible initial states of a lattice and the number of rules, the examination of cellular automata is aimed at dividing the whole problem into smaller subclasses. Wolfram gave the classification of cellular automata [1]. He based his work on the differences in asymptotic behaviour of cellular automata systems. This classification has been specified more precisely by Stauffer and co-workers [2-4]. They found that after a very long time the patterns of homogeneous cellular automata fall into one of six classes of possible final configurations. There are five classes in the above classification where the shapes of patterns stabilize in a fixed way. The rules leading to stabilization can be called *dissipative*. The evolution of stable cellular automata occurs in one of the following ways:

(i) The whole pattern is shifted in one direction; a rule works as a translation by one lattice site. This can be called *fixed point stabilization*.

(ii) The whole pattern is shifted in one direction together with flipping all spins; a rule works as the composition of a translation and a spin value conjugation. This is the *oscillating fixed point* class.

(iii) The spin configurations are repeated periodically and during one time period the pattern is shifted by a spatial period in one direction; a rule works as a periodic transformation which after some (often $L/2$, or L) steps combine to the translation—*limit cycles*.

Dissipative rules can be classified not only according to the pattern shapes but also with respect to the size of a basin attraction [5]. Hence, the idea of this classification is based on the number of initial states which are attracted to the final state.

Let us consider a subclass of rules, $r_s \in R_S$, which are symmetric with respect to up-down symmetry:

$$r_s(-\Theta_i(t)) = 1 - r_s(\Theta_i(t)) \tag{1.3}$$

where $-\Theta_i(t) = (1 - E_i(t), 1 - N_i(t), 1 - W_i(t), 1 - S_i(t))$. Notice that for any rule r_s, there exists an *anti-rule*, r_s^A, defined by the following equation:

$$r_s^A(\Theta_i(t)) = 1 - r_s(\Theta_i(t)). \tag{1.4}$$

Thus, any cellular automaton $[\sigma^A]$ governed by r_s^A evolves in the following way:

$$\sigma_i^A(1) = 1 - \sigma_i(1) \qquad \sigma_i^A(2) = r_s^A(\Theta_i(1)) = 1 - (1 - r_s(\Theta_i(1))) = \sigma_i(2).$$

Therefore after two steps the patterns of both $[\sigma]$ and $[\sigma^A]$ coincide. Hence, if a rule stabilizes the system as a fixed point then the corresponding anti-rule will lead the system to the oscillating fixed point, and reversely (see [6] for details).

In the whole set of $2^{16} = 65\,536$ of possible homogeneous rules on a square lattice, there are 5.7 and 2.8% rules stabilizing as fixed points or oscillations of period 2 (classes 0–4 in the classification of [3, 4]), respectively [2]. The class of symmetric rules consists of $2^8 = 256$, which stands for only 0.39% of all rules. But among them 136 always reach the stabilization and for a further 48 there is a suspicion that their stabilization does not always appear because of the short observation time.

There exist four patterns that play a significant role among all final states obtained in the evolution of symmetric cellular automata. These are as follows:

$$\text{cluster-board} \quad \begin{matrix} 0000 \\ 0000 \\ 0000 \\ 0000 \end{matrix} \quad \text{or} \quad \begin{matrix} 1111 \\ 1111 \\ 1111 \\ 1111 \end{matrix} \tag{1.5}$$

$$\text{chess-board} \quad \begin{array}{c} 0101 \\ 1010 \\ 0101 \\ 1010 \end{array} \tag{1.6}$$

$$\text{line-board} \quad \begin{array}{c} 0000 \\ 1111 \\ 0000 \\ 1111 \end{array} \quad \text{or} \quad \begin{array}{c} 1010 \\ 1010 \\ 1010 \\ 1010 \end{array} \tag{1.7}$$

$$\text{pair-board} \quad \begin{array}{c} 0011 \\ 0110 \\ 1100 \\ 1001 \end{array} \quad \text{or} \quad \begin{array}{c} 0011 \\ 1001 \\ 1100 \\ 0110. \end{array} \tag{1.8}$$

Their meaning follows from the fact that all of them are stable points of the evolution equation (1.2). It means that all patterns are unchangeable by any symmetric rule. Hence, if $r_s \in R_S$ then $[\sigma_0] \equiv r_s[\sigma_0]$ where $[\sigma_0]$ is one of (1.5), (1.6), (1.7) or (1.8), and the equivalence is up to a translation or a translation + conjugation. Of course, there are only a few rules producing such shapes from any random initial state. All others exhibit the property of conserving the above given special shapes only. Notice that a few neighbourhoods result in building the patterns, and because of this they can be uniquely characterized by the function which points out some particular neighbourhoods.

The aim of the present paper is to describe all homogeneous symmetric cellular automata by the distributions of neighbourhoods in the final patterns. Since there exists a unique correspondence between rules and such distributions, we obtain a powerful tool for the further examination of cellular automata.

The paper is organized as follows: section 2 contains the description of computer experiments and results. In section 3 a qualitative explanation of the obtained results is given while section 4 contains final conclusions.

2. Simulation results

Figure 1 presents 16 different configurations ϑ_i corresponding to different states of the four nearest neighbours around an ith spin σ_i on the square lattice (1.1). One can assign a number n ranging from 15 to 0 to configurations as indicated in figure 1. Notice that in pairs $(n, 15-n)$ the configurations $\vartheta(n)$ and $\vartheta(15-n)$ are symmetric with respect to up–down symmetry. A rule is defined as the set of *up* or *down* spin states which are taken by a central spin in the next time step according to the configuration of its neighbours. So each rule can be uniquely characterized by the following number [3]:

$$r = \sum_{n=0}^{15} \sigma_i(\vartheta(n)) 2^n. \tag{2.1}$$

By the property (1.3), the symmetric rules $r_s \in R_S$ act symmetrically on symmetric configurations. Therefore the 16 elements of the set $\{\sigma_i(\vartheta(n))\}$ are not independent.

0	0	1	0	0	
0 0	0 1	0 0	1 0	0 0	
0	0	0	0	1	
15(7)	*14(6)*	*13(5)*	*12(4)*	*11(3)*	
1	0	0	1	1	0
0 1	1 1	0 1	1 0	0 0	1 0
0	0	1	0	1	1
10(2)	*9(1)*	*8(0)*	*7*	*6*	*5*
1	1	0	1	1	
1 1	0 1	1 1	1 0	1 1	
0	1	1	1	1	
4	*3*	*2*	*1*	*0*	

Figure 1. Configurations and their numbers. Numbers in parentheses are for symmetric rules numbers only.

They satisfy the following relation:

$$\sigma_i(\vartheta(n)) = 1 - \sigma_i(\vartheta(15-n)). \tag{2.2}$$

Consequently, instead of 16 variables we deal with 8 independent ones. Therefore, the number of all rules 2^{16} is reduced to $2^8 = 256$. Taking numbers $n = 7, \ldots, 0$ (numbers in brackets in figure 1) the rule description (2.1) can also be reduced to

$$r_s = \sum_{n=0}^{7} \sigma_i(\vartheta(n)) 2^n \tag{2.3}$$

and its uniqueness is still preserved. Notice that the anti-rule's number, r_s^A (1.4), has the following property $r_s^A = 255 - r_s$.

The properties (1.3) and (1.4) allow us to decrease the total number of symmetric rules by half. For the purpose of further consideration attention is focused only on the rules which have the property $\sigma_i(\vartheta(7)) = 0$. Furthermore, 128 such rules can be grouped into 32 classes of equivalence with respect to rotational symmetry (rotation by $\pm\pi/2$, π). Finally, among these 32 classes there are 8 which are equivalent with respect to mirror symmetry to some other classes. It reduces the number of classes to 24. Hence, 24 rules fully represent 256 symmetric rules. All 24 representatives considered are listed in the first column in table 1. The number of rules within a particular class is given in the last column of table 1.

Some properties of the symmetric rules have been presented in [6, 7]. One can find there, for example, a full characterization of the final state of the lattice of homogeneous and symmetric automata in the language of three macroscopic functions:

(i) *magnetization*—the number of spins being in *up* state;

(ii) *activity*—the number of flipping spins in the last time step;

(iii) *time*—the number of computer steps needed to reach the stabilization.

These results give us some hints how the set of symmetric rules can be considered; however, they do not allow an easy explanation of the properties of the system.

For our further consideration one of the above-mentioned functions, namely time, is of importance. Recall that the whole set R_S has been divided there into three parts:

(i) rules which always stabilize the system (dissipative rules): first 12 rules in table 1;

Table 1. The probabilities of finding the $\vartheta(i)$ configuration in a final pattern, $\{p(\vartheta(i)): i = 15, \ldots, 8\}$ with their STD errors. All numbers are in per cent. In the last column is given the number of rules which are equivalent to the one considered. $L = 44$, $P = 0.5$.

Rule number	Configuration $p(\vartheta(15))$	$p(\vartheta(14))$	$p(\vartheta(13))$	$p(\vartheta(12))$	$p(\vartheta(11))$	$p(\vartheta(10))$	$p(\vartheta(9))$	$p(\vartheta(8))$	Number of rules
1	47.9±25	0.23±0.26	0.18±0.45	0.22±0.26	0.18±0.45	0.63±0.60	0.00±0.02	0.68±0.69	8
6	47.0±21	1.37±0.93	0.01±0.04	1.37±0.93	0.01±0.03	0.08±0.27	0.00±0.00	0.09±0.29	8
9	6.3±0.60	6.2±0.37	6.2±0.44	6.3±0.38	6.3±0.38	6.4±0.57	6.3±0.55	6.2±0.54	8
10	12.8±1.6	5.9±0.40	6.0±0.41	5.9±0.34	6.0±0.33	12.8±0.83	0.00±0.00	0.00±0.00	16
11	8.9±0.96	4.8±0.41	8.8±0.39	4.8±0.30	8.9±0.42	7.2±0.51	0.00±0.00	6.8±0.56	8
13	7.0±0.58	8.9±0.42	4.7±0.34	9.0±0.48	4.7±0.36	0.00±0.00	6.9±0.39	8.8±0.53	16
25	0.78±1.1	0.28±0.59	0.25±0.22	0.28±0.59	0.25±0.22	47.5±1.6	0.66±0.60	0.00±0.01	16
27	1.4±1.0	0.03±0.08	0.80±0.69	0.03±0.08	0.80±0.69	47.5±1.6	0.34±0.65	0.00±0.00	16
41	0.00±0.01	0.22±0.21	0.05±0.23	0.23±0.22	0.05±0.23	0.71±0.56	47.8±1.6	0.81±0.62	8
45	0.00±0.00	1.2±1.4	0.00±0.00	1.2±1.4	0.01±0.02	0.28±1.0	47.7±2.1	0.05±0.23	8
114	0.12±0.56	6.2±0.65	6.1±0.42	6.2±0.68	6.1±0.30	0.11±0.55	12.5±1.1	12.6±1.1	16
118	0.00±0.00	4.7±0.37	8.9±0.50	4.7±0.47	8.8±0.54	7.0±0.60	8.8±0.54	7.0±0.36	8
12	11.9±2.9	11.9±2.1	0.53±0.21	11.9±2.1	0.53±0.21	0.52±0.16	12.3±2.4	0.52±0.16	8
14	30.5±12	6.3±1.3	3.5±1.4	6.2±1.3	3.5±1.4	0.12±0.14	0.02±0.04	0.10±0.12	8
115	0.02±0.04	5.9±1.9	3.3±0.73	5.9±1.9	3.3±0.71	0.12±0.12	30.9±2.7	0.13±0.13	8
116	0.56±0.17	0.51±0.14	12.2±0.8	0.54±0.18	12.2±0.7	11.7±1.5	0.53±0.16	11.8±1.3	8
117	0.12±0.15	3.3±0.8	6.0±0.8	3.4±0.9	6.1±0.8	30.7±1.9	0.15±0.13	0.04±0.08	16
42	9.7±1.9	7.7±0.67	4.7±0.50	7.6±0.62	4.7±0.34	5.3±0.52	4.9±0.52	5.4±0.53	8
43	5.9±0.63	6.0±0.47	5.9±0.59	6.0±0.33	6.0±0.47	6.3±0.55	6.5±0.37	6.2±0.58	8
49	5.5±0.89	7.7±0.51	4.7±0.38	7.6±0.44	4.7±0.39	9.3±1.1	5.4±0.45	5.5±0.37	16
51	6.2±0.9	6.3±0.5	6.0±0.3	6.2±0.5	6.1±0.4	6.5±0.5	6.0±0.5	6.1±0.6	16
113	6.4±0.81	6.3±0.45	6.3±0.52	6.3±0.55	6.3±0.47	6.1±0.67	6.3±0.44	6.3±0.57	8
121	5.0±0.67	4.7±0.37	7.3±0.50	4.6±0.44	7.4±0.58	5.3±0.77	9.8±1.4	5.2±0.36	8
124	6.2±0.60	6.2±0.48	6.2±0.33	6.3±0.42	6.1±0.37	6.2±0.56	6.3±0.38	6.3±0.45	8

(ii) rules for which stabilization sometimes occurs: 5 rules in the middle of table 1;

(iii) rules for which stabilization has never been observed: last 7 rules in table 1. Moreover, rules 1 and 6 always give patterns of *clustering* type (1.5), (1.6), rules 25 and 28 always produce patterns of *pair* type (1.7), and rules 41 and 44 of *line* type (1.8) (see [6] for additional details).

The purpose of the present computer experiments is to find distributions of probabilities of all configurations appearing in the final patterns of symmetric and homogeneous cellular automata. The results are collected in tables 1 and 2. The results obtained exhibit a considerable regularity; therefore, apart from the results obtained, their *STD errors* are also included in both tables. Computer experiments are performed for periodic lattices with the size $L = 44$ and with two types of random initial states. Firstly, the probability P of a single spin to point *up* is taken as $P = 0.5$ and therefore all initial configurations are of equal probability (table 1). Secondly, since some rules provide results depending on the initial probability P, (see [6] for magnetization and activity dependence on P), experiments with $P = 0.1$ were also performed (table 2). The results obtained for the latter case of the initial conditions are presented only when they differ more than is allowed by STD errors. One can learn about initial distributions of neighbourhoods by looking at rule 9 in both tables. The special role of this rule will be explained in the last section.

We have to explain the case when the considered rules stabilize the system in a periodic way. In this case the numbers of configurations vary periodically in time.

Table 2. The distribution functions in the case $L=44$ and $P=0.1$ (%).

	Configuration							
Rule	$p(\vartheta(15))$	$p(\vartheta(14))$	$p(\vartheta(13))$	$p(\vartheta(12))$	$p(\vartheta(11))$	$p(\vartheta(10))$	$p(\vartheta(9))$	$p(\vartheta(8))$
number	$p(\vartheta(0))$	$p(\vartheta(1))$	$p(\vartheta(2))$	$p(\vartheta(3))$	$p(\vartheta(4))$	$p(\vartheta(5))$	$p(\vartheta(6))$	$p(\vartheta(7))$
1	100±0.00	0.00±0.00	0.00±0.00	0.00±0.00	0.00±0.00	0.00±0.00	0.00±0.00	0.00±0.00
	0.00±0.00	0.00±0.00	0.00±0.00	0.00±0.00	0.00±0.00	0.00±0.00	0.00±0.00	0.00±0.00
6	100.0±00.00	0.00±0.00	0.00±0.00	0.00±0.00	0.00±0.00	0.00±0.00	0.00±0.00	0.00±0.00
	0.00±0.00	0.00±0.00	0.00±0.00	0.00±0.00	0.00±0.00	0.00±0.00	0.00±0.00	0.00±0.00
9	65.6±0.55	7.2±0.20	7.3±0.31	7.2±0.24	7.3±0.33	0.88±0.22	0.84±0.19	0.77±0.22
	0.02±0.03	0.09±0.07	0.09±0.06	0.09±0.07	0.10±0.07	0.89±0.23	0.82±0.17	0.78±0.21
10	67.3±0.8	6.7±0.34	6.6±0.29	7.2±0.29	7.2±0.23	2.1±0.41	0.00±0.00	0.00±0.00
	0.14±0.10	0.72±0.19	0.71±0.19	0.16±0.10	0.14±0.11	0.10±0.24	0.00±0.00	0.00±0.00
11	67.1±0.7	6.7±0.29	6.9±0.23	6.7±0.30	7.4±0.21	1.3±0.20	0.00±0.00	0.82±0.14
	0.03±0.04	0.06±0.04	0.70±0.20	0.07±0.06	0.23±0.13	0.79±0.12	0.00±0.00	1.3±0.22
13	65.3±0.8	8.0±0.34	7.0±0.32	7.9±0.33	7.1±0.31	0.00±0.00	0.76±0.22	0.93±0.19
	0.00±0.01	0.09±0.07	0.07±0.06	0.15±0.09	0.01±0.03	0.00±0.00	1.6±0.23	1.0±0.19
41	1.3±1.0	7.1±2.5	0.00±0.00	7.1±2.5	0.00±0.00	0.93±0.37	36.8±4.7	0.97±0.37
	0.00±0.00	0.10±0.07	0.00±0.00	0.07±0.06	0.00±0.00	0.95±0.34	33.8±2.7	0.95±0.34
45	1.3±1.1	9.6±2.9	0.00±0.00	9.6±2.9	0.00±0.00	0.00±0.00	34.7±4.5	0.00±0.00
	0.00±0.00	0.46±0.75	0.00±0.00	0.46±0.74	0.00±0.00	0.00±0.00	43.9±1.9	0.00±0.00
114	0.00±0.00	5.9±0.34	8.1±0.48	6.2±0.39	7.9±0.43	0.00±0.00	8.2±0.62	20.1±0.72
	0.00±0.00	3.3±0.40	4.7±0.51	3.1±0.42	4.9±0.57	0.00±0.00	7.9±0.64	19.7±0.85
118	0.00±0.00	1.5±0.25	9.5±0.71	1.6±0.28	9.3±0.75	12.3±0.74	3.6±0.64	12.8±0.76
	0.00±0.00	1.5±0.28	9.0±0.77	1.4±0.23	9.2±0.66	12.4±0.75	3.5±0.61	12.5±0.71
12	64.0±1.6	8.9±0.31	6.8±0.41	8.9±0.31	6.8±0.41	0.00±0.00	0.91±0.34	0.00±0.00
	0.03±0.05	0.28±0.64	0.05±0.06	0.28±0.66	0.05±0.06	0.00±0.02	2.8±0.39	0.00±0.02
14	67.1±1.8	7.3±0.28	7.1±0.29	7.3±0.28	7.1±0.29	0.00±0.00	0.00±0.00	0.00±0.00
	0.39±0.30	1.0±0.44	0.79±0.34	1.0±0.44	0.79±0.34	0.00±0.00	0.00±0.00	0.00±0.00

Since the differences between particular time steps are so small and they fall within the STD error interval, they can be neglected.

The average results obtained for $P=0.5$ follow from at least 50 computer experiments and they are symmetric with respect to up–down symmetry, as expected.

For comparison we also simulate lattices with helical boundary conditions [2] and with different linear sizes $L=43$ and 98. The observed differences between helical and periodic cases do not exceed STD errors. Because of the vertical twist in the helical case, we restrict our search for the dynamics stabilization to the stabilization of the main part of a pattern: that is, without its first and last columns. In some experiments different limit evolutions were observed, depending on different boundary conditions. For example, the limit point stabilization was obtained in one case, and the periodic stabilization with the period equal to the lattice size L in some other cases. It does not affect the distribution of neighbourhoods.

The evolution is stopped when the system stabilizes or when the number of time steps is bigger than 200 in the cases $L=43$, and 44 or 600 for $L=98$.

The case $L=43$ is important because of the fact that some rules never lead to stabilization or stabilize the considered system in another way when the lattice size is an odd number. It refers to five rules for the middle part of table 1 [6]. The results obtained exhibit greater error intervals but the averages for particular neighbourhoods do not differ from the case when L is even.

Results for $L=98$ also do not differ from $L=43$ and 44 but they are more steady. The corresponding STD errors are smaller by half. It was also observed that if any configuration occurred on the $L=44$ lattice with probability smaller than 1%, then in the case $L=98$ it would be observed with the probability reduced by at least half.

One can see that there is a unique correspondence between the distribution of configurations in final patterns and dissipative rules. Any stabilizing rule can be characterized by 16 fixed numbers $\{p(\vartheta(i)),\ i=15,\ldots,0\}$ corresponding to the probability of finding any neighbourhood in the final patterns. Since for some rules the initial probability P influences the final states, the above numbers have to be indexed by P. To be precise, one can introduce a family $\boldsymbol{D}$ of distributions of configurations in final patterns, $D_P^{r_d}=\{p_P^{r_d}(\vartheta(i)):\ i=15,\ldots,0\}$, defined on the set of dissipative and symmetric rules $r_d\in R_d$ and on the interval of initial probabilities $(0,1)$:

$$\boldsymbol{D}: \qquad R_d\times(0,1)\to D_P^{r_d}\,. \tag{2.4}$$

Then, magnetization can be easily expressed by the $\boldsymbol{D}$ function in the following way:

$$M([\sigma])=\frac{L^2}{4}\sum_{i=0}^{15} a_i p_P^{r_d}(\vartheta(i)) \tag{2.5}$$

where a_i are the numbers of *up* states in a $\vartheta(i)$ configuration.

Because of the shape of distribution functions, one can make the general division of the domain R_d into two subsets: rules with sharp peaks: 1, 6, 25, 27, 41, 45 and rules with strong zeros: 10, 11, 13, 114, 118, 12, 14, 115, 116, 117. Rule 9 plays a special role. It conserves any initial pattern.

Since STD errors are small, the domain of $\boldsymbol{D}$ can be extended to the whole set of symmetric rules. The general difference between the stabilizing and non-stabilizing cases consists in the fact that there are neither peaks nor zeros in the distribution of non-stable automata.

Non-stabilizing rules are independent of P (see table 2, and [6]). Notice that if the rule is independent of P, then $\boldsymbol{D}$ becomes the global characterization of the rule attractor.

3. Analysis of stabilities

Instead of considering a rule r as a function of four variables corresponding to the particular neighbours, one can view r as a function over the set of configurations. Then, r is a function of one variable and its domain consists of 16 elements, labelled by numbers in figure 1. Because of the spin values taken by neighbours, the elements of the domain can be grouped as follows:

$\boldsymbol{A}$: all neighbours have the same value: 2 configurations, 15, 0;

$\boldsymbol{B}$: three neighbours have the same value: 8 configurations, $14,\ldots,11,4,\ldots,1$;

$\boldsymbol{C}$: there are two pairs with different values: 6 configurations, $10,\ldots,5$.

From the point of view of a symmetric rule r_s, it is enough to consider half of each group. Thus, we choose as follows:

$$\boldsymbol{A}=\{15\} \qquad \boldsymbol{B}=\{14,\ldots,11\} \qquad \boldsymbol{C}=\{10,9,8\}. \tag{3.1}$$

Hence, any r_s can be defined as a mapping from the union of the separate subdomains $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ to the set of 'actions' which is also divided into three parts, respectively, to the actions on the subdomains:

$$r_s: \qquad \boldsymbol{A} \cup \boldsymbol{B} \cup \boldsymbol{C} \to \left(\begin{pmatrix} F \\ FA \end{pmatrix}, \begin{pmatrix} FF \\ AF \\ S \\ AS \\ \vdots \\ S-E \\ S-N \\ \vdots \end{pmatrix}, \begin{pmatrix} s \\ as \\ \vdots \end{pmatrix} \right). \tag{3.2}$$

The notation introduced above can be easily explained by the following observations. There exist two possible actions $\sigma_i(t+1)$ of a spin σ_i in time $t+1$ in the case when its neighbourhood $\Theta_i(t)$ in time t forms an $\boldsymbol{A}$ configuration: the first consists in following neighbours in a *Ferro* ways,

$$\sigma_i(t+1) = 0 \qquad \text{if } \Theta_i(t) \in \boldsymbol{A} \tag{3.3}$$

and the second one in an *anti-Ferro* way,

$$\sigma_i(t+1) = 1 \qquad \text{if } \Theta_i(t) \in \boldsymbol{A}. \tag{3.4}$$

Let us denote them F and FA, respectively. Notice that r_s with property (3.4) is called an anti-Rule in the previous section.

The set of actions over the $\boldsymbol{B}$ subdomain, $\boldsymbol{B}$ actions, is the greatest one because the number of $\boldsymbol{B}$ elements is the biggest. All $16 = 2^4$ $\boldsymbol{B}$ actions can be defined as:

(i) clustering: 2 actions

FF: $\quad \sigma_i(t+1) = 0 \qquad \text{if } \Theta_i(t) \in \boldsymbol{B}$

AF: $\quad \sigma_i(t+1) = 1 \qquad \text{if } \Theta_i(t) \in \boldsymbol{B}$.

(ii) shifts: 8 actions

S: $\quad \sigma_i(t+1) = S_i(t) \qquad \text{if } \Theta_i(t) \in \boldsymbol{B}$.

E, N, W: as above and the letter denotes the neighbour which is shifted.

AS: $\quad \sigma_i(t+1) = 1 - S_i(t) \qquad \text{if } \Theta_i(t) \in \boldsymbol{B}$.

AE, AN, AW: as above and letters denote both the spin value conjugation and the neighbour which is shifted.

(iii) alternative shifts: 6 actions

$S-E$: $\quad \sigma_i(t+1) = \max(S_i(t), E_i(t)) \qquad \text{if } \Theta_i(t) \in \boldsymbol{B}$.

$S-N$, $S-W$, $E-W$, $E-N$, $N-W$: as above with active neighbours denoted by the letters.

The last part of r_s is responsible for the rule action when the neighbourhood of a spin is $\boldsymbol{C}$ type, $\boldsymbol{C}$ action. One can see that it has to be one from the shifts listed as follows:

s:	$\sigma_i(t+1)=S_i(t)$ if $\Theta_i(t)\in \boldsymbol{C}$.
e, n, w:	as above and the letter denotes the neighbour which is shifted.
as:	$\sigma_i(t+1)=1-S_i(t)$ if $\Theta_i(t)\in \boldsymbol{C}$.
ae, an, aw:	as above and letters denotes both the spin value conjugation and the neighbour which is shifted.

The relations between the numbers of the rules (2.3) and actions (3.2) are presented in table 1 are given as

$$\begin{aligned}
&1=(F, FF, s) \quad && 6=(F, FF, as) \quad && 9=(F, S, s)\\
&10=(F, S, w) && 11=(F, S, an) && 13=(F, S, aw)\\
&25=(F, W-S, s) && 27=(F, W-S, an) && 41=(F, S-N, s)\\
&45=(F, S-N, aw) && 114=(F, AS, w) && 118=(F, AS, as)\\
&12=(F, S, n) && 14=(F, S, as) && 115=(F, AS, an)\\
&116=(F, AS, n) && 117=(F, AS, aw) && 42=(F, S-N, w)\\
&43=(F, S-N, an) && 49=(F, W-N, s) && 51=(F, W-N, an)\\
&113=(F, AS, s) && 121=(F, AF, s) && 124=(F, AF, as).
\end{aligned} \tag{3.5}$$

The actions of any rule on the separated parts of domain are clearly seen from equations (3.5). When one considers a lattice where spins with $\boldsymbol{A}$, $\boldsymbol{B}$ or $\boldsymbol{C}$ neighbourhoods are separated from each other, then one can observe evolution as a product:

$$r(\boldsymbol{A}\cup\boldsymbol{B}\cup\boldsymbol{C})=r(\boldsymbol{A})\times r(\boldsymbol{B})\times r(\boldsymbol{C}). \tag{3.6}$$

On a lattice the only way to see separated actions is to create a lattice with all neighbourhoods belonging to one subdomain. One can check that patterns (1.5)–(1.8) fulfil this condition. On random initial states the elements from different subdomains are randomly mixed. If $P=0.5$ then half of the configurations belong to the $\boldsymbol{B}$ subdomain, and therefore any action on the $\boldsymbol{B}$ elements is essential for the whole evolution of automata. This action dominates over other partial movements in the following sense: *automata reach the stabilization almost always on a pattern where a rule can work as a translation in the direction which agrees with the* **B** *action.* There are two exceptions from the above observation: when there exists a strong contradiction between $\boldsymbol{B}$ and $\boldsymbol{C}$ actions (rules (F, S, n) and (F, AS, n)) and when the $\boldsymbol{B}$ action does not determine any direction (rules (F, FF) type). In the first case the evolution has the property that all changes made during one period combines with the translation by the number of period length in the direction fixed by the $\boldsymbol{B}$ action. In the second case, the fixed point stabilization is rarely not a translation in one direction, but there are a few parts in a final pattern shifted in different directions [7].

The role of the $\boldsymbol{B}$ neighbourhood is easy to see when the $\boldsymbol{B}$ action is a shift (see table 3). Rules $9=(F, S, s)$ and its anti-rule $246=(FA, AS, as)$ play a special role. Rule 9 is a translation from South, hence rule 246 is a composition of a translation from South and a spin conjugation, and any initial pattern is its own attractor. The evolution with a rule r_s which has the same $\boldsymbol{B}$ action but acts differently on the $\boldsymbol{C}$ subdomain, can be described as the elimination of neighbourhoods on which the rule r_s differs from one of the pair 9 and 246. The smaller number of differences determines towards

Table 3. Differences between the rules and the property of a rule of leading to the stalization. For comparison with theoretical predictions, results are given referring to the length of time needed by the automata to reach the stabilization and the type of the stabilization observed. $L = 44$, $P = 0.5$, and time of observation does not exceed 400 steps.

Rule number and action	Odd config. to SHIFT	Odd config. to SHIFT+CONJ.	$\langle T\rangle$ to stabilize	Probability and limit
$9 = (F, S, s)$	None	$15, \ldots, 8$	$\langle T\rangle = 0$	1 point
$11 = (F, S, an)$	9	$15, \ldots, 10, 8$	$\langle T\rangle = 6$	1 point
$13 = (F, S, aw)$	10	$15, \ldots, 11, 9, 8$	$\langle T\rangle = 6$	1 point
$10 = (F, S, w)$	9, 8	$15, \ldots, 10$	$\langle T\rangle = 11$	1 point
$12 = (F, S, n)$	10, 9	$15, \ldots, 11, 8$	$\langle T\rangle = 50$	1 point/period
$14 = (F, S, as)$	10, 9, 8	$15, \ldots, 11$	$\langle T\rangle = 206$	0.5 point
$113 = (F, AS, s)$	14, 13, 12, 11	15, 10, 9, 8	Never	0
$115 = (F, AS, an)$	$14, \ldots, 11, 9$	15, 10, 8	$\langle T\rangle = 250$	0.26 oscillation
$117 = (F, AS, aw)$	$14, \ldots, 11, 10$	15, 9, 8	$\langle T\rangle = 206$	0.53 oscillation
$116 = (F, AS, n)$	$14, \ldots, 11, 10, 8$	15, 9	$\langle T\rangle = 50$	1 oscillation/period
$114 = (F, AS, e)$	$14, \ldots, 11, 9, 8$	15, 10	$\langle T\rangle = 12$	1 oscillation
$118 = (F, AS, as)$	$14, \ldots, 8$	15	$\langle T\rangle = 7$	1 oscillation
$246 = (FA, AS, as)$	$15, \ldots, 8$	None	$\langle T\rangle = 0$	1 oscillation

which evolution the rule r_s leads. When the same numbers of differences appear, case $113 = (F, AS, s)$, the stabilization cannot be reached. The conflict in the directions of shifts over the $\boldsymbol{B}$ and $\boldsymbol{C}$ subdomains is also reflected in the increase of the length of time needed to reach the stabilization.

There is also a reaction from the $\boldsymbol{C}$ action on the $\boldsymbol{B}$ subdomain. According to this process one obtains the same probability of finding $\boldsymbol{B}$ configurations having the same vertical (14, 12) and horizontal (13, 11) neighbours. The strongest interaction is observed when the shifts have opposite directions, as in rules $12 = (F, S, n)$ and $116 = (F, AS, n)$. The favoured pair is the one in which $\boldsymbol{B}$ actions N and S or N and AS agree. The differences in probabilities between pairs also depend on P (compare tables 1 and 2). Other $\boldsymbol{C}$ shifts w, e, s, do not change the initial distribution of the $\boldsymbol{B}$ neighbourhoods. They are neutral. The influence of the an $\boldsymbol{C}$ action can be explained in the same way as for n, by comparing $\boldsymbol{B}$ actions S with AN, and AS with AN. The rest of the $\boldsymbol{C}$ actions as, ae, aw, favour the other pair more than an action does.

If the action over $\boldsymbol{B}$ is not of the shift type, the evolution does not only mean elimination of some $\boldsymbol{A}$ or $\boldsymbol{C}$ neighbourhoods, but it also works on converting $\boldsymbol{B}$ neighbourhoods into one from the list (1.5)-(1.8). Such a double effect needs more time than the destroying process. But the stabilization at one of the patterns from (1.5)-(1.8) is only observed if the $\boldsymbol{C}$ action does not oppose the $\boldsymbol{B}$ action. It means that rules $25 = (F, W-S, s)$, $27 = (F, W-S, an)$, $41 = (F, S-N, s)$ and $45 = (F, S-N, aw)$ always lead to stabilization (in a time shorter than 100 steps if $L = 44$) as a translation with one from directions given by the $\boldsymbol{B}$ action. On the other hand, rules $42 = (F, S-N, w)$, $43 = (F, S-N, an)$, $49 = (F, W-N, s)$ and $51 = (F, W-N, an)$ never cause stabilization in our experiments. Notice that stabilization is reached on the pattern where there are only such $\boldsymbol{C}$ neighbourhoods which have the property $W_i = S_i$ in case of rules 25 and 26 and $S_i = N_i$ in cases 41 and 45.

The action FF over the $\boldsymbol{B}$ subdomain destroys $\boldsymbol{B}$ neighbourhoods, independently of the $\boldsymbol{C}$ shift, but also converts them into $\boldsymbol{A}$ neighbourhoods. In this case the $\boldsymbol{B}$ action does not fix the translation direction and therefore the direction depends on the $\boldsymbol{C}$ action but it is not fixed [6, 7]. It rarely happens that a final pattern consists of a few rectangular parts, on which the evolution is a translation but the directions of shifts in different parts are different. The rule with the $\boldsymbol{B}$ action, AF as opposed to FF, cannot give a stable solution.

4. Conclusions

There exists a powerful characterization of dissipative symmetric and homogeneous automata by the distribution of neighbourhoods in the final patterns. The attracting patterns attainable by the automata are firmly determined in this picture if the initial states of automata are taken randomly. The distribution function $\boldsymbol{D}$, given in equation (2.4), as the macroscopic function can give answers to some global questions (for example, it can be used to compute magnetization (2.5)). Moreover, it also gives some new hints concerning the old problem of classification of cellular automata. The final patterns can be viewed not only as a mixture of zeros and ones but as well defined structures which are conserved in time.

The properties of non-stabilizing rules can be expressed by the neighbourhood distribution function as well. Although the automata are not stable, the possible pattern configurations passed during the evolution are greatly restricted.

Since the distribution function $\boldsymbol{D}$ also reflects the individual properties of rules, it can be used as a tool for understanding the nature of cellular automata.

Notice that there is no problem introducing the neighbourhood counters into the computer program because the neighbourhoods have to be recognized during the evolution of homogeneous cellular automata.

The structure of the rules introduced via the discussed actions makes it possible to give a satisfactory explanation not only of the obtained neighbourhood distributions in the final patterns but also allows elucidation of such properties of the automata as: whether or not the stabilization is reached, the length of time needed to stabilize the system, and even type of stabilization.

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